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COMMENT

Fermion determinants and the Berry phase

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Abstract. Following an observation that certain fermion operators are parametrisable by a holonomy variable by which they may be 'rotated' into a 'free' Dirac operator, we show that their determinants are given by their Berry phases.

Considerable interest has been focused during the past decade on the role of topological reasoning in physics. Investigations on anomalies in theories involving fermions is a case in point (Treiman *et al* 1985). In this respect it has been noted that certain classes of fermion operators can be constructed, the determinants of which may be calculated through the chiral anomaly (Alvarez 1984, Nepomechie 1985). The Wess-Zumino topological term arises naturally in this analysis.

In a parallel development various authors have recently drawn attention to the Berry holonomy phase and its connection with anomalies is under active scrutiny (Berry 1984, Jackiw 1988, Shapere and Wilczek 1989). Rigorous studies of the holonomy phase by Simon (1983) and by Kiritsis (1987) have provided criteria regarding the possibility of observing them in specific systems. That the Berry phase is related to the Wess-Zumino term has been shown by Niemi and Semenoff (1985, 1986). It is then interesting to consider whether fermion determinants may also be calculated via the Berry phase. A further reason for this is the recent finding that the Berry phase emerges from an expansion of the effective action for a spin-$\frac{1}{2}$ system in an external field (Düsedau 1988).

We show in this comment that the Berry phase may be used for evaluating a class of fermion determinants and we apply it to a simple paradigm example. By introducing the idea of homotopic paths we find that it is possible to think of 'rotating' fermion operators into 'free' Dirac operators, thereby relating the action to Berry's phase. Homotopic paths and Berry's phase have been studied before (Nelson and Alvarez-Gaumé 1985). In our development we will also give an expression for the change in the path integral measure brought about by an adiabatic closed-loop excursion of the system. Although well known for the Wess-Zumino term (D'Hoker and Farhi 1984, Fujikawa 1986) there is probably no explicit statement of it for the Berry phase in the manner described herein.

We start by considering the Berry phase for a system described by the Dirac equation for a minimally coupled fermion (in the $A^0=0$ gauge)

$$(\mathrm{i}\partial_t + H)\psi(\boldsymbol{x}, t) = \lambda\psi(\boldsymbol{x}, t) \tag{1}$$

where the Hamiltonian H is initially time independent. Suppose now that H evolves adiabatically so that at each instant thereafter the system remains an eigenstate of the instantaneous Dirac operator. If H returns to its original expression in a time interval

T, the excursion of the system may be pictured as transport around a closed path in a parameter space, which we call R space. As pointed out by Berry (1984) the wavefunction of the rth eigenstate acquires in the process a non-integrable phase $\gamma_r(T)$,

$$\gamma_r(T) = \mathrm{i} \int_0^T \mathrm{d}t \left\langle r, t \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| r, t \right\rangle \tag{2}$$

aside from the dynamical one. To be specific let us assume that the adiabatic change is brought about by a unitary operator $U(\boldsymbol{R}(t))$ which 'rotates' the wavefunction ψ to N,

$$N(\boldsymbol{x}(t)) = U(t)\psi(\boldsymbol{x}) \tag{3}$$

where we write $U(t)$ for $U(\boldsymbol{R}(t))$ and suppress time in $\psi(\boldsymbol{x}, t)$.

In the path integral formation we are interested in the $T \to \infty$ limit of

$$\int [D\psi][D\psi^\dagger] \exp\left\{\mathrm{i} \int_0^T \mathrm{d}t\, \psi^\dagger(\mathrm{i}\partial_t + H)\psi\right\}. \tag{4}$$

Under (3) the path measure undergoes the change

$$[D\psi][D\psi^\dagger] \to J[DN][DN^\dagger] \tag{5}$$

where J is the Jacobian. Following Fujikawa (1986) we expand ψ and $\psi^\dagger$ as

$$\psi(\boldsymbol{x}(t)) = \sum_n a_n \varphi_n(\boldsymbol{x}(t)) \qquad \psi^\dagger(\boldsymbol{x}(t)) = \sum_n \varphi_n(\boldsymbol{x}(t)) b_n^\dagger \tag{6}$$

where a_n and $b_n^\dagger$ are Grassmann numbers. The eigenfunctions φ_n satisfy

$$\tilde{H}(t)\varphi_n(\boldsymbol{x}(t)) = \lambda_n(t)\varphi_n(\boldsymbol{x}(t)) \tag{7}$$

with $\tilde{H} = U^\dagger H U$.

By expressing the path measure in the form $\Pi_{m,n}\, \mathrm{d}b_m^\dagger\, \mathrm{d}a_n$ we find that (3) leads us to

$$\prod_{m,n} \mathrm{d}b_m^\dagger\, \mathrm{d}a_n = (\det C_{mn})^2 \prod_{m,n} \mathrm{d}b_m'^\dagger\, \mathrm{d}a_n' \tag{8}$$

where $b_m'^\dagger$ and a_n' are the expansion coefficients for $N^\dagger$ and N and $c_{mn} \equiv \langle \varphi_m, U, \varphi_n \rangle$. The determinant is dealt with by introducing (Gamboa-Saravi *et al* 1981)

$$B_{pm}(t, T) \equiv \langle \varphi_p(\boldsymbol{x}(t+T)), U(T)\varphi_m(\boldsymbol{x}(t))\rangle \tag{9}$$

which satisfies $B(t, T+\delta) = B(t+T, \delta)\ B(t, T)$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln B(0, t) = \frac{1}{\delta t} \ln \det B(t, \delta t).$$

From the expansion

$$\varphi_n(\boldsymbol{x}(t+\delta t)) = \varphi_n(\boldsymbol{x}(t)) + \delta t \sum_{p \neq n} b_{np}\varphi_p(\boldsymbol{x}(t))$$

we have

$$\det\langle \varphi(\boldsymbol{x}(t))\rangle, U(t)\varphi(\boldsymbol{x}, 0)\rangle = \exp \oint \mathrm{tr}\langle \varphi | \nabla\alpha \cdot T | \varphi \rangle\, \mathrm{d}\boldsymbol{R} \tag{10}$$

where we put $U = e^{\alpha \cdot T}$ (T are generators). The right-hand side is unity. Going back to the determinant for c_{mn}

$$\det(c_{mn}) = \det\langle\varphi(\boldsymbol{x}(0))|U(T)\varphi(\boldsymbol{x},(0))\rangle$$

with $\boldsymbol{R}(T) = \boldsymbol{R}(0)$, we subdivide the time interval T into smaller segments and write

$$\det(c_{mn}) = \lim_{N\to\infty} \prod_{m=1}^{N} \langle\varphi(\boldsymbol{x}(t_m))|\varphi(\boldsymbol{x}(t_{m+1}))\rangle \tag{11}$$

with $t_1 = 0$ and $t_N = T$. With the approximation

$$\langle\varphi(\boldsymbol{x}(t_m))|\varphi(x(t_{m+1}))\rangle \simeq 1 + \left\langle\varphi\left|\frac{\partial}{\partial \boldsymbol{R}}\right|\varphi\right\rangle \cdot \Delta\boldsymbol{R} \simeq \exp\left[-\mathrm{i}\left\langle\varphi\left|\mathrm{i}\frac{\partial}{\partial \boldsymbol{R}_i}\right|\varphi\right\rangle\right] \mathrm{d}R_i \tag{12}$$

we have finally

$$\det(c_{mn}) = e^{-\mathrm{i}\Gamma} \tag{13}$$

where

$$\Gamma = \mathrm{i}\oint_C \mathrm{tr}\left\langle\varphi\left|\frac{\partial}{\partial \boldsymbol{R}}\right|\varphi\right\rangle \cdot \mathrm{d}\boldsymbol{R} \tag{14}$$

is just the Berry phase (Kuratsuji and Iida 1985). Thus the Jacobian is

$$J = e^{-2\mathrm{i}\Gamma}. \tag{15}$$

Our next task is the evaluation of (4). Suppose that we can write the action

$$I[\psi, \psi^\dagger, H_\tau(t)] \equiv \mathrm{Tr}\, \psi^\dagger \mathrm{i}(\partial_t + H_\tau(t))\psi \tag{16}$$

(integration implied) in such a way that H is diagonalised by a unitary matrix $U_\tau(t)$,

$$H_{\mathrm{diag}} = U_\tau^\dagger(t) H_\tau(t) U_\tau(t). \tag{17}$$

In the above we have introduced two variables: t is time and τ ($0 \leqslant \tau \leqslant \pi$) is a homotopy variable which parametrises the choice of closed path followed in the adiabatic evolution of H. In line with (3) we define the wavefunction ϕ

$$\psi = U_\tau(t)\phi. \tag{18}$$

We may arrange H such that when $\tau = 0$, $U_\tau(t) =$ identity and H_{diag} is a constant matrix. Referring to this as the free Dirac problem, $I[\psi, \psi^\dagger, H_0(t)]$ is then the free Dirac action. We now compare the change in the action when two homotopic paths labelled by τ and $\tau - \delta\tau$ are traversed. A quick calculation shows that

$$I[\psi, \psi^\dagger, H_\tau(t)] = I[\psi', \psi'^\dagger, H_{\tau-\delta\tau}(t)] + \delta\gamma \tag{19}$$

where

$$\psi' = U_{\tau-\delta\tau}(t)\phi \tag{20}$$

and $\delta\gamma$ is the change in the Berry phase as one traverses the loops τ and $\tau - \delta\tau$. The fermion determinant

$$\det D \equiv e^{\mathrm{i}\Gamma[H_\tau]} \equiv \int [D\psi][D\psi^\dagger]\, e^{\mathrm{i}I[\psi,\psi^\dagger,H_\tau(t))]} \tag{21}$$

together with (15) lead us to

$$-\mathrm{i}\frac{\delta\Gamma[H_\tau]}{\delta\tau} = \delta\gamma. \tag{22}$$

This result may be compared with that of Nepomechie (1985). From our discussion we see that τ may be considered as a variable describing the 'rotation' of H from H_{diag} to some general matrix. Properties of (22) will be presented in a future work; an application to a simple model will be useful in exposing its utility.

We consider the model of Gozzi and Thacker (1987):

$$L = \frac{i}{2}\,\psi_k\dot{\psi}_k + \frac{i}{2}\,\psi_l\hat{B}_{lm}\psi_m \qquad \hat{B}_{lm} = B_k\varepsilon_{klm}. \tag{23}$$

They have diagonalised the matrix $\hat{B}$ and found

$$\hat{B}_{\text{diag}} = \begin{pmatrix} 0 & & \\ & -\mathrm{i}B & \\ & & \mathrm{i}B \end{pmatrix} = U^{\dagger}_{\tau=1}\hat{B}U_{\tau=1}$$

$B = (B_kB_k)^{1/2}$, where we assume that the configuration of (23) for $\hat{B}$ corresponds to $\tau = 1$ and B_{diag} to $\tau = 0$. The matrix U is given in Gozzi and Thacker (1987). The corresponding wavefunctions ϕ of (18) are just their normal coordinates, namely

$$\phi_1 = \text{constant} \qquad \phi_2 \propto \mathrm{e}^{\mathrm{i}Bt} \qquad \phi_3 \propto \mathrm{e}^{-\mathrm{i}Bt}.$$

Berry's phase is found from (2), i.e. the expectation of $U^{\dagger}\dot{U} = \dot{B}_kU^{\dagger}(\partial U/\partial B_k)$. One has that the non-vanishing results are $\delta\gamma = \pm\mathrm{i}\,\dot{\phi}\cos\theta$, the plus sign being for ϕ_3 and minus for ϕ_2. (Here θ, ϕ are the polar angles in B space.) Then from (22) if we assume, as a first approximation, that $\delta\tau = \mathrm{d}\theta$, we have $\delta\Gamma = \delta\Omega$ (Ω = solid angle) which is the first-order result of Düsedau (1988). The determinant will be $\exp[\pm\mathrm{i}(B+\Omega)]$ in this approximation.

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